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Critical Temperature in Bosonic Gases

Luiz Carlos Costa Neto

Abstract— A Bose-Einstein condensate (BEC) is a state of matter of a dilute gas of bosons cooled to absolute zero (0 K or -273.15 ºC) [1]. Under such conditions, the bosons occupy the lowest quantum state, at which point quantum effects become apparent on a macroscopic scale. These effects are called macroscopic quantum phenomena. The extreme cold caused the individual atoms to condense into a "superatom" that behave as a single entity. Recently, researchers at JILA, a joint program of NIST and the University of Colorado at Boulder, recently announced that they created a Bose-Einstein Condensate, predicted almost 90 years ago by Albert Einstein and Indian physicist Satyendra Nath Bose, within a range of temperatures not predicted to the original theory. This paper provides a method to calculate the critical temperature for a bosonic gas to achieve the conditions observed in a Bose-Einstein Condensate.

Index Terms— Bose-Einstein condensate, bosonic gas, critical temperature, macroscopic quantum phenomena, states of matter, quantum state.

1 Introduction

A Bose-Einstein condensate (BEC) is a state of matter of a dilute gas of bosons cooled to temperatures very near absolute zero (0 K or -273.15 ºC). Under such conditions, a large fraction of the bosons occupy the lowest quantum state, at which point quantum effects become apparent on a macroscopic scale. These effects are called macroscopic quantum phenomena.

The Bose-Einstein Condensate state of matter was first predicted by Satyendra Nath Bose and Albert Einstein in 1924-25. Bose first sent a paper to Einstein on the quantum statistics of light quanta (now called photons). Einstein was impressed, translated the paper himself from English to German and submitted it for Bose to the *Zeitschrift für Physik*, which published it. Einstein then extended Bose's ideas to material particles (or matter) in two other papers [2]. The result of the efforts of Bose and Einstein is the concept of a Bose gas, governed by Bose-Einstein statistics, which describes the statistical distribution of identical particles with integer spin, now known as bosons. Bosonic particles, which include the photon as well as atoms such as helium-4, are allowed to share quantum states with each other. Einstein demonstrated that cooling bosonic atoms to absolute zero would cause them to fall (or “condense”) into the lowest accessible quantum state, resulting in a new form of matter.

In 1938 Fritz London proposed BEC as a mechanism for superfluidity in 4-He and superconductivity [4], [5].

3 Critical Temperature

In the first gaseous condensate was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST–JILA lab, using a gas of rubidium atoms cooled to 170 nanokelvin (nK). For their achievements Cornell, Wieman, and Wolfgang Ketterle at MIT received the 2001 Nobel Prize in Physics. In November 2010 the first photon BEC was observed [3].

This transition to BEC occurs below a critical temperature, which for a uniform three-dimensional gas consisting of non-interacting particles with no apparent internal degrees of freedom is given by:

$$T_c = \left(\frac{\eta}{\zeta\left(\frac{3}{2}\right)} \right)^{\frac{2}{3}} \frac{2\pi\hbar^2}{m k_b}$$

where:

T_c is the critical temperature;
η is the particle density;
m is the mass per boson;
$\hbar$ is the reduced Planck constant;
k_b is the Boltzmann constant; and
ζ is the Riemann zeta function

4 Einstein's Argument

Consider a collection of N noninteracting particles, which can each be in one of two quantum states, $|0\rangle$ and $|1\rangle$. If the two states are equal in energy, each different configuration is equally likely.

If we can tell which particle is which, there are 2^N different configurations, since each particle can be in $|0\rangle$ or $|1\rangle$ independently. In almost all of the configurations, about half the particles are in $|0\rangle$ and the other half in $|1\rangle$. The balance is a statistical effect: the number of configurations is largest when the particles are divided equally.

If the particles are indistinguishable, however, there are only $N+1$ different configurations. If there are K particles in state $|1\rangle$, there are $N-K$ particles in state $|0\rangle$. Whether any particular particle is in state $|0\rangle$ or in state $|1\rangle$ cannot be determined, so each value of K determines a unique quantum state for the whole system. If all these states are equally likely, there is no statistical spreading out; it is just as likely for all the particles to sit in $|0\rangle$ as for the particles to be split half and half.

Suppose now that the energy of state $|1\rangle$ is slightly greater than the energy of state $|0\rangle$ by an amount E. At temperature T, a particle will have a lesser probability to be in state $|0\rangle$ by $e^{\left(-\frac{E}{kT}\right)}$. In the distinguishable case, the particle distribution will be biased slightly towards state $|0\rangle$ and the distribution will be slightly different from half and half. But in the indistinguishable case, since there is no statistical pressure toward equal numbers, the most likely outcome is that most of the particles will collapse into state $|0\rangle$.

In the distinguishable case, for large N, the fraction in state $|0\rangle$ can be computed. It is the same as flipping a coin with probability proportional to $p = e^{\left(\frac{-E}{T}\right)}$ to land tails. The probability to land heads is $\frac{1}{1+p}$, which is a smooth function of p and thus of the energy.

In the indistinguishable case, each value of K is a single state, which has its own separate Boltzmann probability. So the probability distribution is exponential:

$$P(k) = C e^{\left(\frac{-KE}{T}\right)} = C P^{k}$$

For large N, the normalization constant C is $(1-p)$. The expected total number of particles not in the lowest energy state, in the limit that $N \to \infty$, is equal to:

$$\sum_{n>0} Cnp^{n} = \frac{(p)}{(1-p)}$$

It does not grow when N is large, it just approaches a constant. This will be a negligible fraction of the total number of particles. So a collection of enough Bose particles in thermal equilibrium will mostly be in the ground state, with only a few in any excited state, no matter how small the energy difference.

Consider now a gas of particles, which can be in different momentum states labeled $|k\rangle$. If the number of particles is less than the number of thermally accessible states, for high temperatures and low densities, the particles will all be in different states. In this limit the gas is classical. As the density increases or the temperature decreases, the number of accessible states per particle becomes smaller, and at some point more particles will be forced into a single state than the maximum allowed for that state by statistical weighting. From this point on, any extra particle added will go into the ground state.

To calculate the transition temperature at any density, integrate over all momentum states the expression for maximum number of excited particles $\frac{(p)}{(1-p)}$:

$$N = V \int \frac{d^3k}{(2\pi)^3} \frac{p(k)}{1-p(k)} = V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\frac{k^2}{2mT}} - 1}$$

So

$$p(k) = e^{\frac{-k^2}{2mT}}$$

When the integral is evaluated with the factors of k_b and $\hbar$ restored by dimensional analysis, it gives the critical temperature formula of the preceding section. Therefore, this integral defines the critical temperature and particle number corresponding to the conditions of negligible chemical potential. In Bose–Einstein statistics distribution, μ is actually still nonzero for Bose-Enstein Condensates; however, μ is less than the ground state energy. Except when specifically talking about the ground state, μ can consequently be approximated for most energy or momentum states as $\mu \approx 0$.

5 Conclusion

This work proposes a new way to calculate critical temperatures for Bose-Einstein Condensates, and it may help experimental cientists to increase the precision of their experiments in superconductivity and superfluidity.

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